## <u>Fourier series - tasks</u> ( II- part)

**Primer 4.**

Function $f(x)=|x|-1$ developed in Fourier series on the interval $[-1,1]$ and find sum $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$.

**<u>Solution:</u>**

As is $f(-x)=|-x|-1=|x|-1=f(x)$ we conclude that the function is even .

We use the formula:

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l})$$

$$a_0=\frac{1}{l}\int_{-l}^{l}f(x)dx \qquad a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}dx \qquad b_n=0$$

$$a_0=\frac{1}{l}\int_{-l}^{l}f(x)dx=\frac{1}{1}\int_{-1}^{1}(x-1)dx=2\int_{0}^{1}(x-1)dx=2\left(\frac{x^2}{2}-x\right)/_0^1=-1$$

$$a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}dx=\frac{1}{1}\int_{-1}^{1}(x-1)\cos\frac{n\pi x}{1}dx=2\int_{0}^{1}(x-1)\cos n\pi x dx$$

As always, we will solve this integral with the help of partial integration:

$$\int(x-1)\cos n\pi x dx=\begin{vmatrix}x-1=u & \cos n\pi x dx=dv\\ dx=du & \frac{1}{n\pi}\sin n\pi x=v\end{vmatrix}=(x-1)\cdot\frac{1}{n\pi}\sin n\pi x-\int\frac{1}{n\pi}\sin n\pi x dx=$$

$$=\frac{(x-1)\sin n\pi x}{n\pi}-\frac{1}{n\pi}\int\sin n\pi x dx==\frac{(x-1)\sin n\pi x}{n\pi}+\frac{1}{n\pi}\frac{1}{n\pi}\cos n\pi x$$

$$=\frac{(x-1)\sin n\pi x}{n\pi}+\frac{1}{(n\pi)^2}\cos n\pi x$$

Now get back to put boundaries:

$$a_n=2\int_{0}^{1}(x-1)\cos n\pi x dx=2\left(\frac{(x-1)\sin n\pi x}{n\pi}+\frac{1}{(n\pi)^2}\cos n\pi x\right)/_0^1=$$

$$=2\left[\left(\frac{(1-1)\sin n\pi 1}{n\pi}+\frac{1}{(n\pi)^2}\cos n\pi 1\right)-\left(\frac{(0-1)\sin n\pi 0}{n\pi}+\frac{1}{(n\pi)^2}\cos n\pi 0\right)\right]$$

$$=2\left(\frac{1}{(n\pi)^2}\cos n\pi-\frac{1}{(n\pi)^2}\right)=\frac{2}{(n\pi)^2}(\cos n\pi-1)=\frac{2}{(n\pi)^2}\left((-1)^n-1\right)$$

Similar to the previous examples, think of even and odd *n,* and is:

$$a_n = \begin{cases} 0, & n = 2k \\ -\frac{4}{(n\pi)^2}, & n = 2k-1 \end{cases}$$

Now we're going to formula:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l})$$

$$f(x) = |x| - 1 = \frac{1}{2}(-1) + \sum_{k=1}^{\infty}\frac{-4}{(2k-1)^2\pi^2}\cos\frac{(2k-1)\pi x}{1}$$

$$\boxed{|x| - 1 = -\frac{1}{2} - \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos(2k-1)\pi x}{(2k-1)^2}}$$

Let's look at requested sum : $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$ . We see that in our series we have to put x = 0 :

$$|0| - 1 = -\frac{1}{2} - \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos(2k-1)\pi 0}{(2k-1)^2}$$

$$-1 = -\frac{1}{2} - \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}$$

$$\frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{1}{2} \rightarrow \boxed{\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{\pi^2}{8}}$$

**Example 5.**

Function *f(x) = x-2* developed in Fourier series on the interval *[1,3]* .

*Solution:*

We must use the formula:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos\frac{2n\pi x}{b-a} + b_n \sin\frac{2n\pi x}{b-a})$$

$$a_0 = \frac{2}{b-a}\int_a^b f(x)dx \qquad a_n = \frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}dx \qquad b_n = \frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi x}{b-a}dx$$

So,we have:

$$a_0 = \frac{2}{3-1}\int_1^3 (x-2)dx = \left(\frac{x^2}{2} - 2x\right)\Big/\begin{matrix}3\\1\end{matrix} = \left(\frac{3^2}{2} - 6\right) - \left(\frac{1^2}{2} - 2\right) = 0$$

$$a_n = \frac{2}{3-1}\int_1^3 (x-2)\cos\frac{2n\pi x}{3-1}dx =$$

$$= \int_1^3 (x-2)\cos n\pi x dx$$

To solve this first without borders:

$$\int (x-2)\cos n\pi x dx = \begin{vmatrix} x-2=u & \cos n\pi x dx = dv \\ dx = du & \frac{1}{n\pi}\sin n\pi x = v \end{vmatrix} = (x-2)\cdot\frac{1}{n\pi}\sin n\pi x - \int \frac{1}{n\pi}\sin n\pi x dx =$$

$$= \frac{(x-2)\sin n\pi x}{n\pi} - \frac{1}{n\pi}\int \sin n\pi x dx = = \frac{(x-2)\sin n\pi x}{n\pi} + \frac{1}{n\pi}\frac{1}{n\pi}\cos n\pi x$$

$$= \frac{(x-2)\sin n\pi x}{n\pi} + \frac{1}{(n\pi)^2}\cos n\pi x$$

$$a_n = \int_1^3 (x-2)\cos n\pi x dx = \left(\frac{(x-2)\sin n\pi x}{n\pi} + \frac{1}{(n\pi)^2}\cos n\pi x\right)\Big/\begin{matrix}3\\1\end{matrix} =$$

$$= \left(\frac{(3-2)\sin n\pi 3}{n\pi} + \frac{1}{(n\pi)^2}\cos n\pi 3\right) - \left(\frac{(1-2)\sin n\pi 1}{n\pi} + \frac{1}{(n\pi)^2}\cos n\pi 1\right) =$$

$$= \frac{\sin 3n\pi}{n\pi} + \frac{1}{(n\pi)^2}\cos 3n\pi + \frac{\sin n\pi}{n\pi} - \frac{1}{(n\pi)^2}\cos n\pi$$

$$= \frac{1}{(n\pi)^2}\cos 3n\pi - \frac{1}{(n\pi)^2}\cos n\pi$$

$$= \frac{1}{(n\pi)^2}[\cos 3n\pi - \cos n\pi]$$

Remember trigonometric formula: $\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$, if we use it:

$$a_n = \frac{1}{(n\pi)^2}[\cos 3n\pi - \cos n\pi] = \frac{1}{(n\pi)^2}[-2\sin 2n\pi\cdot\sin n\pi] = 0 \rightarrow \boxed{a_n = 0}$$

More to find:

$$b_n = \frac{2}{3-1}\int_1^3 (x-2)\sin\frac{2n\pi x}{3-1}dx = \int_1^3 (x-2)\sin n\pi x dx$$

$$\int (x-2)\sin n\pi x dx = \begin{vmatrix} x-2=u & \sin n\pi x dx = dv \\ dx = du & -\frac{1}{n\pi}\cos n\pi x = v \end{vmatrix} = -(x-2)\cdot\frac{1}{n\pi}\cos n\pi x + \int \frac{1}{n\pi}\cos n\pi x dx =$$

$$= \frac{-(x-2)\cos n\pi x}{n\pi} + \frac{1}{n\pi}\int \cos n\pi x dx = = \frac{-(x-2)\cos n\pi x}{n\pi} + \frac{1}{n\pi}\frac{1}{n\pi}\sin n\pi x$$

$$= \frac{-(x-2)\cos n\pi x}{n\pi} + \frac{1}{(n\pi)^2}\sin n\pi x$$

To insert boundaries:

$$b_n = \int_1^3 (x-2)\sin n\pi x dx = \left(\frac{-(x-2)\cos n\pi x}{n\pi} + \frac{1}{(n\pi)^2}\sin n\pi x\right) / \begin{matrix} 3 \\ 1 \end{matrix} =$$

$$\left(\frac{-(3-2)\cos n\pi 3}{n\pi} + \frac{1}{(n\pi)^2}\sin n\pi 3\right) - \left(\frac{-(1-2)\cos n\pi x}{n\pi} + \frac{1}{(n\pi)^2}\sin n\pi 1\right) =$$

$$-\frac{\cos n\pi 3}{n\pi} - \frac{\cos n\pi x}{n\pi} = -\frac{1}{n\pi}\left(\cos 3n\pi + \cos n\pi\right)$$

Again must use formula for $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

$$b_n = -\frac{1}{n\pi}\left(\cos 3n\pi + \cos n\pi\right) = -\frac{1}{n\pi}2\underbrace{\boxed{\cos 2n\pi}}_{\text{this is 1}}\cos n\pi = -\frac{2}{n\pi}(-1)^n = \frac{2}{n\pi}(-1)^{n+1}$$

$$\boxed{b_n = (-1)^{n+1}\frac{2}{n\pi}}$$

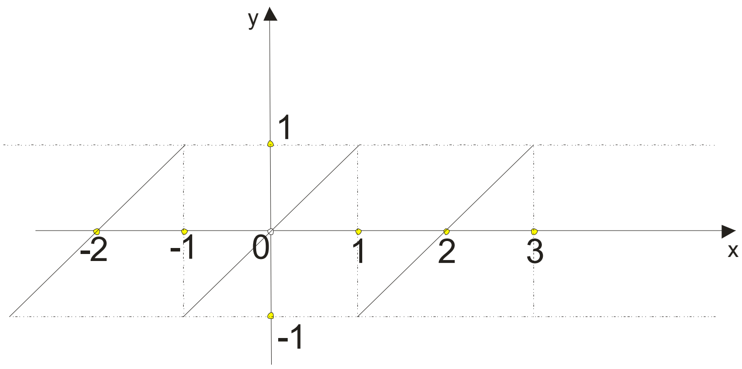
Now we're going to formula:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos\frac{2n\pi x}{b-a} + b_n \sin\frac{2n\pi x}{b-a})$$

Beware:

f(1-0)=1 , f(1+0)=1 and f(3-0)=1, f(3+0)=-1

See image:




then is:

$$S(x)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n\pi}\sin\frac{2n\pi x}{3-1}$$

$$S(x)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n\pi}\sin n\pi x=\{\begin{matrix} x-2, & x\in(1,3) \\ 0, & x\in\{1,3\}\end{matrix}$$

**Example 6.**

Function $f(x)=\left\{\begin{matrix} x, & x\in(0,1) \\ 2-x, & x\in[1,2]\end{matrix}\right\}$ develop in series by:

a) by sine

b) by cosine

Solution:

a)

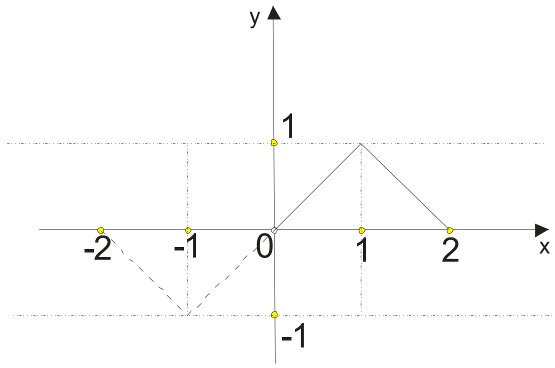
$$f(x)=\left\{\begin{matrix} x, & x\in(0,1) \\ 2-x, & x\in[1,2]\end{matrix}\right\}$$

In order to develop this function by sinus, we must make it to be **an odd** function

It will do the following:

$$F(x)=\left\{\begin{matrix} 2-x, & x\in[1,2] \\ x, & x\in(-1,1) \\ -2-x, & x\in[-2,-1]\end{matrix}\right\}$$

Let's look at how this function looks in the picture:




Of course that here $a_0$ and $a_n$ are equal to zero and we ask: $b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx = \frac{2}{l}\int_{0}^{l} f(x)\sin\frac{n\pi x}{l}dx = \frac{2}{2}\int_{0}^{2} f(x)\sin\frac{n\pi x}{2}dx = \int_{0}^{2} f(x)\sin\frac{n\pi x}{2}dx$$

$$b_n = \int_{0}^{2} f(x)\sin\frac{n\pi x}{2}dx = \int_{0}^{1} x\sin\frac{n\pi x}{2}dx + \int_{1}^{2}(2-x)\sin\frac{n\pi x}{2}dx$$

After solving these integrals, the method of partial integration, in a similar way as in the previous examples we have

$$b_n = \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}$$

We are considering how behave $\sin\frac{n\pi}{2}$. We know that n takes the values 1,2,3 ….

For n = 1 $\sin\frac{n\pi}{2} = \sin\frac{\pi}{2} = 1$

For n = 2 $\sin\frac{n\pi}{2} = \sin\frac{2\pi}{2} = 0$

For n = 3 $\sin\frac{n\pi}{2} = \sin\frac{3\pi}{2} = -1$

For n = 4 $\sin\frac{n\pi}{2} = \sin\frac{4\pi}{2} = 0$

For n = 5 $\sin\frac{n\pi}{2} = \sin\frac{5\pi}{2} = 1$

For n = 6 $\sin\frac{n\pi}{2} = \sin\frac{6\pi}{2} = 0$

*etc.*

Therefore, we conclude: $b_n = \begin{cases} 0, & n = 2k \\ (-1)^k \dfrac{8}{(2k+1)^2\pi^2}, & n = 2k+1 \end{cases}$ k=0,1,2,3…….

And is:

$$f(x) = \frac{8}{\pi^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\sin\frac{(2k+1)\pi x}{2}, for\ x \in (0,2]$$

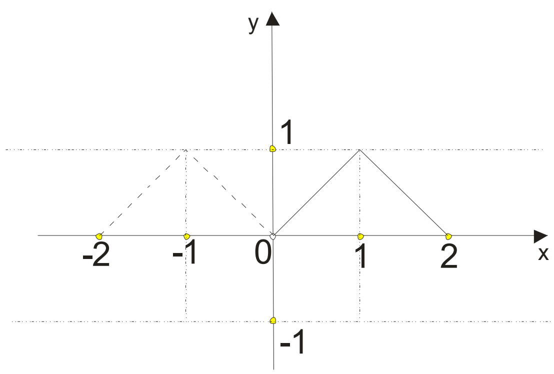
$$F(x) = \frac{8}{\pi^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\sin\frac{(2k+1)\pi x}{2}, for\ x \in [-2,2]$$

b)

For the development function by cosine we have to do as follows:

$$F(x) = \begin{cases} x+2, & x \in [-2,-1] \\ |x|, & x \in (-1,1) \\ x-2, & x \in [1,2] \end{cases}$$

The function is shown in the following picture :




Of course, it is now $b_n = 0$ and : $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$ $a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx = \frac{2}{l}\int_{0}^{l} f(x)dx$$
$$a_0 = \frac{2}{2}\int_{0}^{2} f(x)dx = \int_{0}^{2} f(x)dx = \int_{0}^{1} xdx + \int_{1}^{2}(2-x)dx = 1$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx = \frac{2}{l}\int_{0}^{l} f(x)\cos\frac{n\pi x}{l}dx$$
$$a_n = \frac{2}{2}\int_{0}^{2} f(x)\cos\frac{n\pi x}{2}dx = \int_{0}^{2} f(x)\cos\frac{n\pi x}{2}dx = \int_{0}^{1} x\cos\frac{n\pi x}{2}dx + \int_{1}^{2}(2-x)\cos\frac{n\pi x}{2}dx$$

Solve these integrals and we get:

$$a_n = \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2}(1+\cos n\pi)$$

Let's consider how it behaves term $\cos\frac{n\pi}{2}$ for different n .

For n=1 $\cos\frac{n\pi}{2} = \cos\frac{\pi}{2} = 0$

For n=2 $\cos\frac{n\pi}{2} = \cos\frac{2\pi}{2} = -1$

For n=3 $\cos\frac{n\pi}{2} = \cos\frac{3\pi}{2} = 0$

For n=4 $\cos\frac{n\pi}{2} = \cos\frac{4\pi}{2} = 1$

*etc.*

So, if $n=2k+1$ , then $a_n = 0$

Let's look at an even n, but the form $n = 4k$ or $n = 4k+2$ for $k = 0,1,2,3$ ... ....

*For n=4k*

$$a_n = \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2}(1+\cos n\pi)$$
$$a_{4k} = \frac{8}{(4k)^2\pi^2}\cos\frac{4k\pi}{2} - \frac{4}{(4k)^2\pi^2}(1+\cos 4k\pi) = \frac{8}{16k^2\pi^2}\cos 2k\pi - \frac{4}{16k^2\pi^2}(1+1)$$
$$= \frac{8}{16k^2\pi^2}\cos 2k\pi - \frac{8}{16k^2\pi^2}\cos 2k\pi = 0$$

*For n=4k+2*

$$a_n = \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2}(1+\cos n\pi)$$

$$a_{4k} = \frac{8}{(4k+2)^2\pi^2}\cos\frac{(4k+2)\pi}{2} - \frac{4}{(4k+2)^2\pi^2}(1+\cos(4k+2)\pi)$$

$$= \frac{\not{8}}{\not{4}(2k+1)^2\pi^2}\cos\frac{\not{2}(2k+1)\pi}{\not{2}} - \frac{\not{4}}{\not{4}(2k+1)^2\pi^2}(1+\cos 2\pi(2k+1))$$

$$= \frac{2}{(2k+1)^2\pi^2}\underbrace{\cos(2k+1)\pi}_{\text{This is } -1} - \frac{1}{(2k+1)^2\pi^2}(1+1)$$

$$= -\frac{2}{(2k+1)^2\pi^2} - \frac{2}{(2k+1)^2\pi^2} = \boxed{-\frac{4}{(2k+1)^2\pi^2}}$$

Finally we have:

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{\cos(2k+1)\pi x}{(2k+1)^2}, \quad x \in (0,2] \quad \text{and} \qquad F(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{\cos(2k+1)\pi x}{(2k+1)^2}, \quad x \in [-2,2]$$